A Low-Complexity Projection Algorithm for ADMM-Based LP Decoding

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Maximum-Likelihood (ML) Decoding

Theorem (Feldman et. al)

ML decoding is equivalent to an integer program, i. e.,

$$x^* = \arg\max_{x \in C} \Pr_{noise}\left[\tilde{x} \text{ received } | x \text{ transmitted }\right]$$

can be rewritten as

$$\begin{aligned} \min \quad & \lambda^T x \\ s.t. \quad & Ax \leq b \\ & 0 \leq x \leq 1 \\ & x \in \mathbb{Z}^n \end{aligned}$$

where $\lambda_j = \ln \frac{Pr[\tilde{y}_j | y_j = 0]}{Pr[\tilde{y}_j | y_j = 1]}$ *are the so-called log-likelihood ratios (LLR)*

LP Decoding

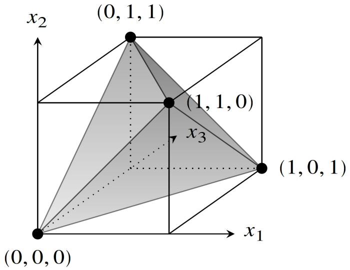
- Relax integrality $\rightarrow$ LP decoding problem:

$$\begin{aligned} \min \quad & \lambda^T x \\ \text{s.t. } & Ax \leq b \\ & 0 \leq x \leq 1 \end{aligned}$$

- Can be solved by Alternating Direction Method of Multipliers (ADMM)
- Main effort of ADMM: projections onto parity polytopes P_{d_j}

$$\mathcal{P}_{d_j} := \text{conv}\{x \in \{0,1\}^{d_j} : \sum_{i=1}^{d_j} x_i \text{ is even}\}.$$

New Projection Algorithm - Geometrical idea



Figure[1] of $\mathcal{P}_3 := \text{conv}\{x \in \{0,1\}^3 : x_1 + x_2 + x_3 \text{ is even}\}$

[1] M. Helmling, S. Ruzika, and A. Tanatmis., "Mathematical programming decoding of binary linear codes: theory and algorithms" in IEEE Transactions on Information Theory 58.7 (July 2012), pp. 4753–4769

New Projection Algorithm - Geometrical idea

$$\mathcal{P}_3 := \mathrm{conv}\{x \in \{0,1\}^3 : x_1 + x_2 + x_3 \text{ is even}\}$$

- Example: $\hat{x} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{11}{4} \end{pmatrix}$
- Goal: Compute projection $\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \Pi_{\mathcal{P}_3} \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{11}{4} \end{pmatrix}$

New Projection Algorithm - Geometrical idea

$$\mathcal{P}_3 := \text{conv}\{x \in \{0,1\}^3 : x_1 + x_2 + x_3 \text{ is even}\}$$

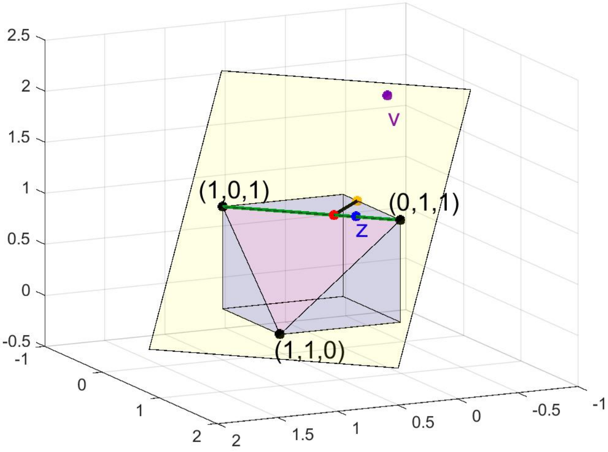
- First idea: project $\hat{x}$ on hypercube $[0,1]^3 \rightarrow$ easy
- $\left(\Pi_{[0,1]^3}(\hat{x})\right)_i = \begin{cases} 1 & \text{if } \hat{x}_i > 1 \\ 0 & \text{if } \hat{x}_i < 0 \\ \hat{x}_i & \text{else} \end{cases} \quad \text{for } i = 1,2,3$
- Simple case: $\Pi_{[0,1]^3}(\hat{x})$ lies in $\mathcal{P}_3 \Rightarrow \Pi_{\mathcal{P}_3}(\hat{x}) = \Pi_{[0,1]^3}(\hat{x})$ $\rightarrow$ finished ✓
- Difficult case: $\Pi_{[0,1]^3}(\hat{x})$ is not in $\mathcal{P}_3$

New Projection Algorithm - Geometrical idea

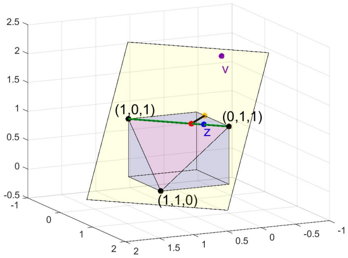
- Example: $\Pi_{[0,1]^3}(\hat{x}) = \Pi_{[0,1]^3}\begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{11}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$
- Cut-search algorithm outputs (potentially) violated inequality $x_1 + x_2 + x_3 \leq 2$
- $\begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$ violates $x_1 + x_2 + x_3 \leq 2 \Rightarrow \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$ is not in $\mathcal{P}_3$
- Zhang & Siegel: $\Pi_{\mathcal{P}_3}(\hat{x})$ lies on facet

$$\{x \in [0,1]^3 : x_1 + x_2 + x_3 = 2\}$$

New Projection Algorithm - Geometrical idea

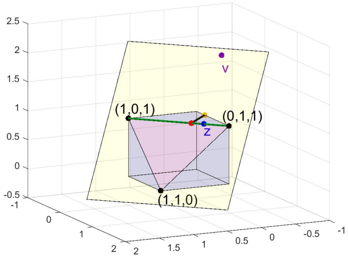


New Projection Algorithm - Geometrical idea



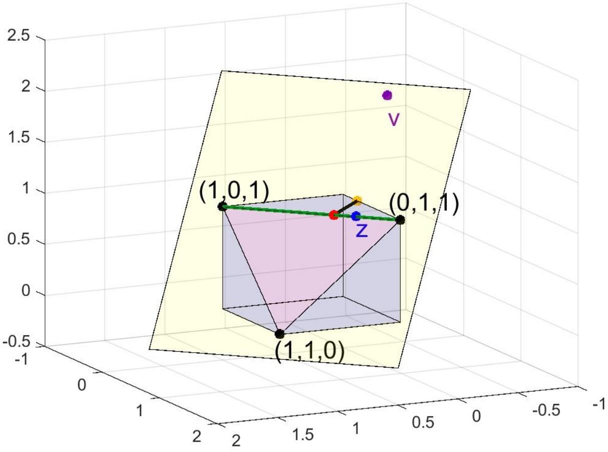
- Need to project on $[0,1]^3 \cap \{x : x_1 + x_2 + x_3 = 2\}$
- Idea:
  - Projection onto cube $[0,1]^3$ is simple
  - Projection onto plane $x_1 + x_2 + x_3 = 2$ is simple

New Projection Algorithm - Geometrical idea

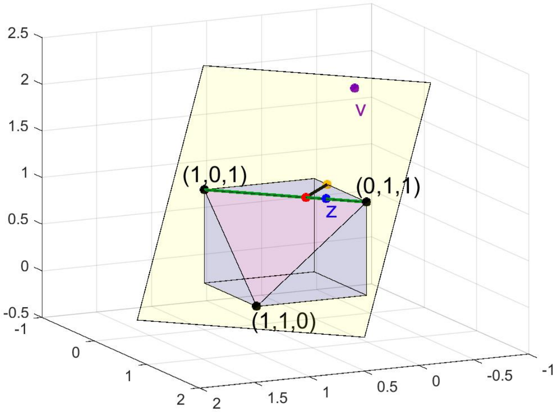


- Project $\hat{x} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{11}{4} \end{pmatrix}$ onto plane $x_1 + x_2 + x_3 = 2$
- $\Rightarrow v = \Pi_{\{x_1+x_2+x_3=2\}} \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{11}{4} \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{4} \\ 2 \end{pmatrix}$

New Projection Algorithm - Geometrical idea

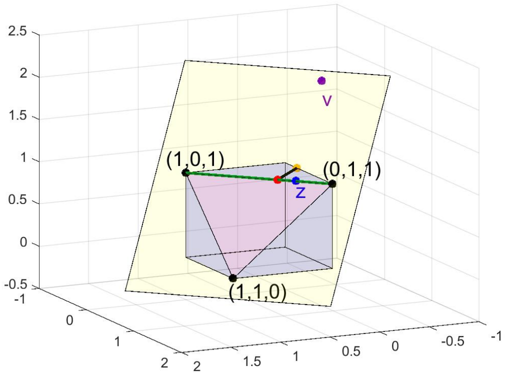


New Projection Algorithm - Geometrical idea



- Next idea: project v onto $[0,1]^3$ (yellow point) $\rightarrow$ bad idea
- Projecting it to the facet does not help (red point)

New Projection Algorithm - Geometrical idea



$$\Rightarrow \text{Wanted projection } \Pi_{\mathcal{P}_3}\begin{pmatrix}\frac{1}{2}\\1\\\frac{11}{4}\end{pmatrix}=\begin{pmatrix}z_1\\z_2\\1\end{pmatrix}$$

New Projection Algorithm - Geometrical idea

- Wanted projection $\Pi_{\mathcal{P}_3}\begin{pmatrix}\frac{1}{2}\\ 1\\ \frac{11}{4}\end{pmatrix} = \begin{pmatrix}z_1\\ z_2\\ 1\end{pmatrix}$
- How to compute z_1 and z_2? $\to$ recursion
- One can show:

$$\left\{\begin{pmatrix}x_1\\ x_2\end{pmatrix} : \begin{pmatrix}x_1\\ x_2\\ 1\end{pmatrix} \in \mathcal{P}_3\right\} = \underbrace{\operatorname{conv}\left\{\begin{pmatrix}x_1\\ x_2\end{pmatrix} \in \{0,1\}^2 : x_1 + x_2 \text{ is odd}\right\}}_{=:\mathcal{P}_{2,\mathrm{odd}}}$$

New Projection Algorithm - Geometrical idea

- $\rightarrow$ Solve smaller-dimensional projection problem
- Recursive problem $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \Pi_{\mathcal{P}_{2,\text{odd}}} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$
- First step of projection: Check whether $\Pi_{[0,1]^2} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \in \mathcal{P}_{2,\text{odd}}$

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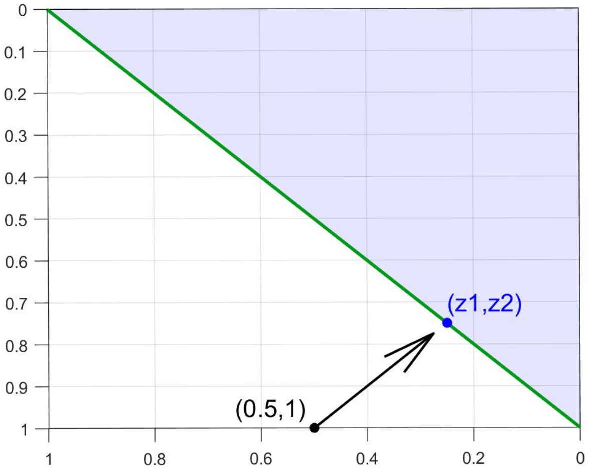
- We showed: Check is never fulfilled in recursion
  $\rightarrow$ omit check in recursion
- Second step: compute inequality to violated facet
- We showed:
  - Use old inequality $x_1 + x_2 + x_3 \leq 2$
  - Insert computed components (here: $x_3 = 1$)
  - $\rightarrow$ Current violated inequality: $x_1 + x_2 \leq 1$

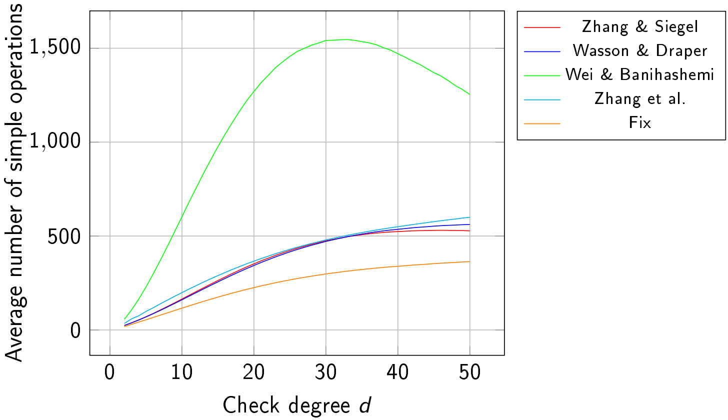
New Projection Algorithm - Geometrical idea



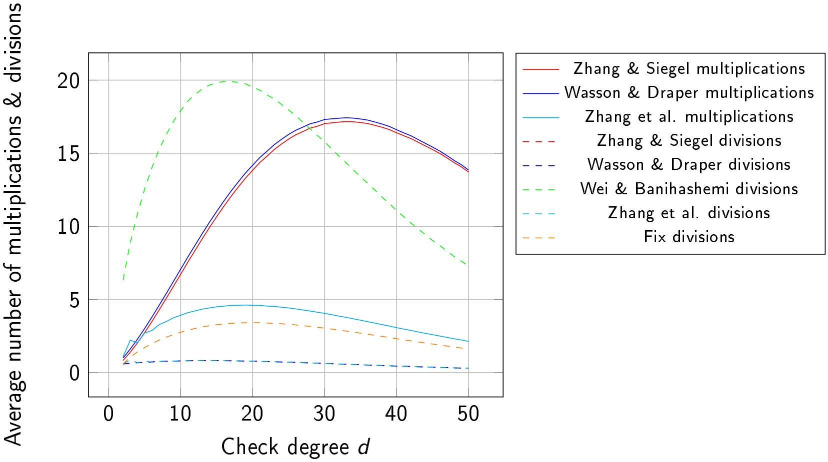
New Projection Algorithm - Geometrical idea

- Recursive problem $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \Pi_{\mathcal{P}_{2,\text{odd}}} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$
- $\Rightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \Pi_{\mathcal{P}_{3,\text{even}}} \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{11}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \\ 1 \end{pmatrix}$

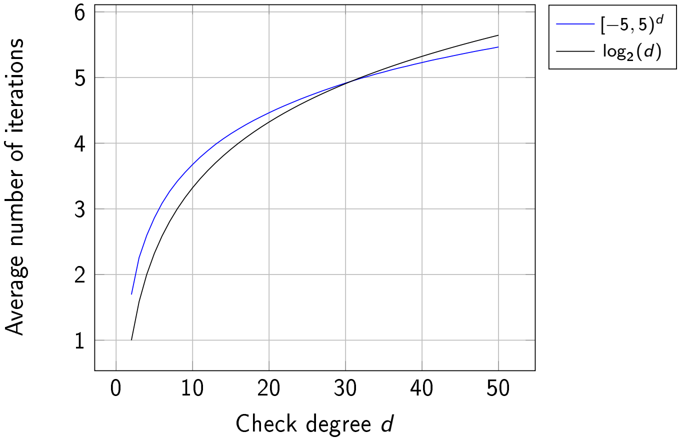
Random vectors from $[-5, 5)^d$



Random vectors from $[-5, 5)^d$



Number of iterations in the difficult case $\Pi_{[0,1]^d}(x) \notin \mathcal{P}_d$



Summary

- New projection algorithm onto parity polytope
- New recursive structure of parity polytope
- Worst-case complexity: $\mathcal{O}(d^2)$
- Conjecture for average-case complexity: $\mathcal{O}(d)$
- 37 % less operations
- No sorting operation required
- Suitable for hardware implementation

Thank you for your attention!