Spatially-Coupled LDPC Codes with Random Access

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Motivation - ECC for Data-Storage

An inherent conflict for data-storage ECC:

No retransmissions $\Longrightarrow$ very strong ECCs are needed.

Strong ECCs $\Longrightarrow$ large blocks/high complexity $\Longrightarrow$ slower read access.

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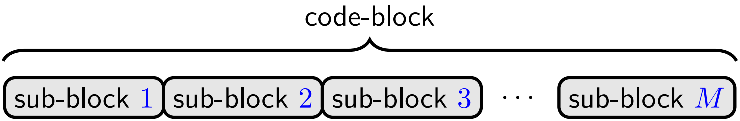
An inherent conflict for data-storage ECC:

No retransmissions $\implies$ very strong ECCs are needed.

Strong ECCs $\implies$ large blocks/high complexity $\implies$ slower read access.

- Extreme error events are rare.

Solution: *random-access* codes [**ISIT'18+arXiv**: random-access **LDPC**].



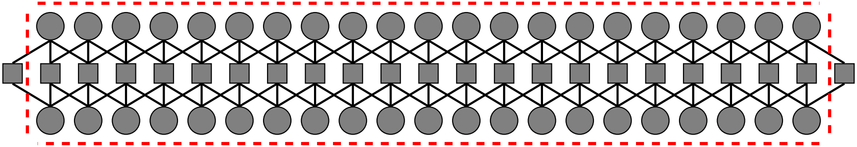
- Fast sub-block read access.
- A "safety net" code-block for increased reliability.

This talk: random-access spatially-coupled LDPC codes!

Prior Work

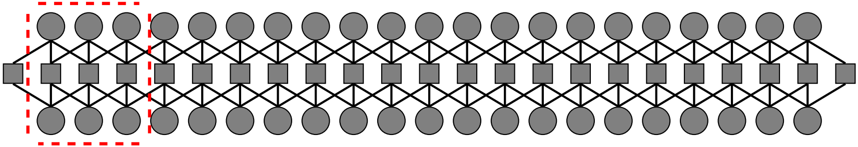
- ER, Cassuto, 2018: **random-access LDPC codes**.
- Cassuto, Hemo, Puchinger, Bossert, 2017: **RS-based random-access codes**.
- Li, Lin, Abdel-Ghaffar, Ryan, Costello 2017: **globally coupled LDPC codes**.
- Mitchell, Lentmaier, Costello, 2015: **protograph-based SC-LDPC codes**.

The $(3, 6)$ spatially-coupled (SC) LDPC protograph:



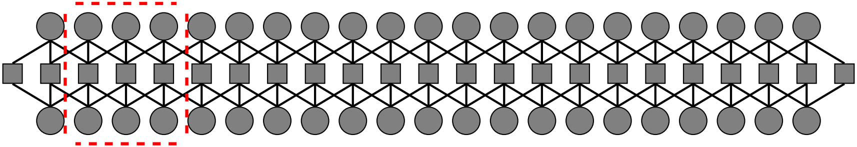
- It is known how to decode **the entire chain**.

The $(3, 6)$ spatially-coupled (SC) LDPC protograph:



- It is known how to decode it **from left to right**: pipelined, windowed.

The $(3, 6)$ spatially-coupled (SC) LDPC protograph:



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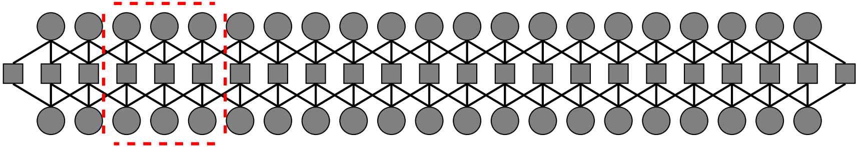

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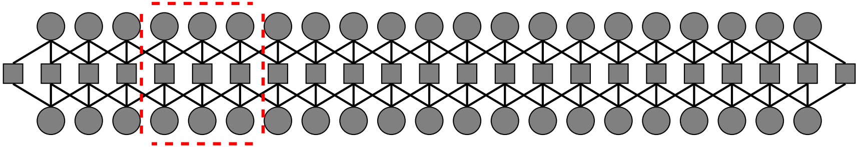

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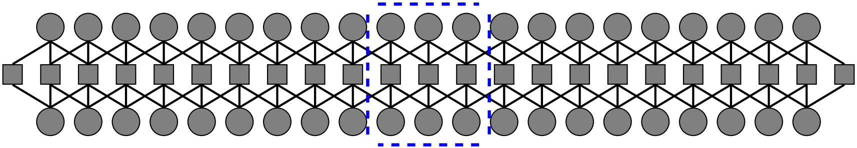


Can we decode an arbitrary sub-block?

Our Contribution

We prove

Existing SC-LDPC codes **do not enable sub-block random access**.

We introduce

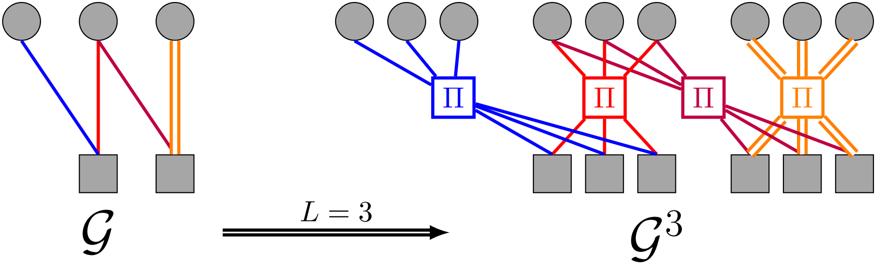
SC-LDPC codes with random access: **SC-LDPCL**.

We suggest and analyze

a novel decoder for SC-LDPCL codes: **semi-global decoding**.

Protograph-Based LDPC Codes

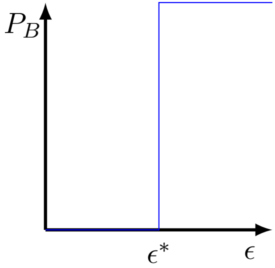
- LDPC protograph: a small bipartite graph $\mathcal{G} = (\mathcal{V} \cup \mathcal{C}, \mathcal{E})$.
- The bi-adjacency matrix of $\mathcal{G}$: $H_{\mathcal{G}} \in \mathbb{N}^{|\mathcal{C}| \times |\mathcal{V}|}$.
- Lifting the **protograph** generates a **Tanner graph** $\mathcal{G}^L$.
  - L is the **lifting parameter**.
  - Lifting introduces randomization.



Lifting Asymptotics

As $L \to \infty$, one can apply density-evolution on $\mathcal{G}$ to analyze Belief Propagation on $\mathcal{G}^L$.

- BEC(ϵ): there exists ϵ^* such that BP will succeed iff $\epsilon < \epsilon^*$.
- ϵ^* is a property of the protograph $\mathcal{G}$.



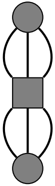
Protograph-Based SC-LDPC Codes:

- Couple regular LDPC protographs by cut and paste:
  - B: a bi-adjacency matrix representing an LDPC protograph.
  - $\{B_t\}_{t=1}^{T}$: matrices such that $\sum_{t=1}^{T} B_t = B$.

$$H_{\mathcal{G}} = \begin{pmatrix} B_1 & & 0 & \\ \vdots & B_1 & & \\ B_T & \vdots & \ddots & \\ & B_T & \ddots & B_1 \\ & & \ddots & \vdots \\ & 0 & & B_T \end{pmatrix}$$

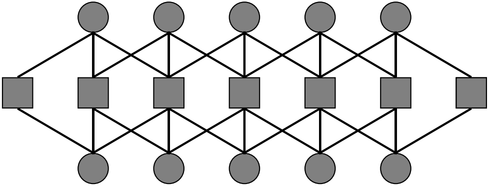
Graphic Example

Regular $(3,6)$ protograph



$B = \begin{pmatrix} 3 & 3 \end{pmatrix}$

Coupled protograph $\mathcal{G}$

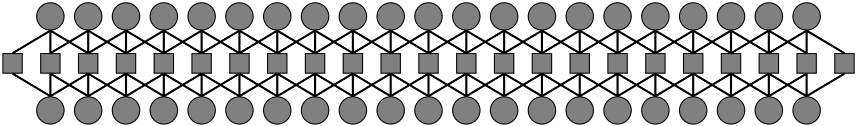


$B_1 = B_2 = B_3 = \begin{pmatrix} 1 & 1 \end{pmatrix}$

$$H_{\mathcal{G}} = \begin{pmatrix} 1&1&0&0&0&0&0&0&0&0 \\ 1&1&1&1&0&0&0&0&0&0 \\ 1&1&1&1&1&1&0&0&0&0 \\ 0&0&1&1&1&1&1&1&0&0 \\ 0&0&0&0&1&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1 \end{pmatrix}$$

Locality in SC-LDPC Codes:

- SC-LDPC codes: long chains are needed to diminish rate-loss.
    - Very large code-block sizes.
- Sub-block access is very valuable.
- Existing SC-LDPC codes do not enable sub-block random access.



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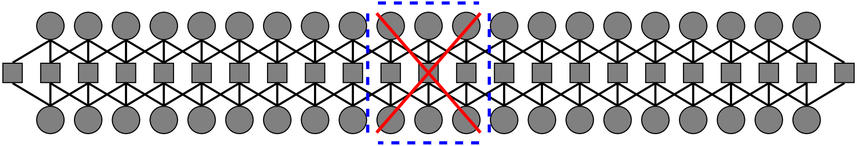

Theorem

Let $\mathcal{G} = (\mathcal{V} \cup \mathcal{C}, \mathcal{E})$ be a SC-LDPC protograph such that for every $t \in \{1, 2, \ldots, T\}$, there are no 2 all-ones rows in B_t, and let $\{\mathcal{V}_m\}_{m=1}^{M}$ be a partition of $\mathcal{V}$ into M sub-blocks. Then, $\epsilon_m^* = 0$, where ϵ_m^* is the asymptotic threshold of sub-block m.

$$H_{\mathcal{G}} = \begin{pmatrix} B_1 & & 0 & \\ \vdots & B_1 & & \\ B_T & \vdots & \ddots & \\ & B_T & \ddots & B_1 \\ & & \ddots & \vdots \\ & 0 & & B_T \end{pmatrix}$$

Proof Outline

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- We can use checks that are connected **only** to $\mathcal{V}_m$ (local checks).
  - $\underline{H_i} \triangleq \{j \in \mathcal{V} \colon H_{i,j} = 1\}, \quad i \in \mathcal{C}$.
  - $\mathcal{C}_m \triangleq \{i \in \mathcal{C} \colon \underline{H_i} \subseteq \mathcal{V}_m\}, \quad m \in \{1, 2, \ldots, M\}$.

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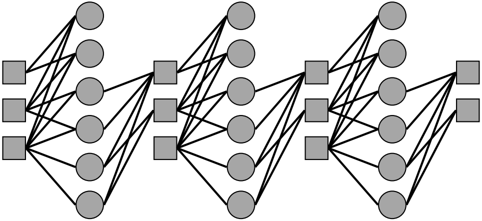
- Local decoding: we effectively work with the local protograph $\mathcal{G}_m = (\mathcal{V}_m \cup \mathcal{C}_m, \mathcal{E}_m)$.



$(3, 6)$ SC-LDPC protograph $\mathcal{G}$

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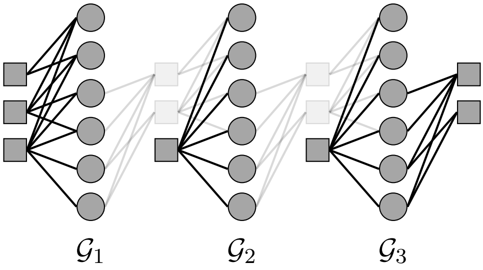
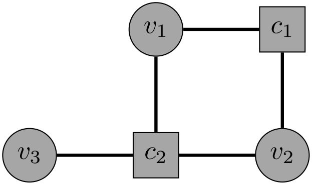


Proof Outline (Cont'd)

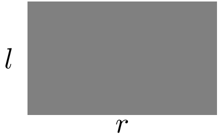
- Let H_m be the bi-adjacency matrix representing $\mathcal{G}_m$, $1 \leq m \leq M$.
- For every $t \in \{1, 2, \ldots, T\}$ there are no 2 all-ones rows in B_t, so one of the following must hold:
  1. H_m has a zero column.
  2. H_m has a column sub-matrix in which all variables are of degree one.
  3. H_m has a column sub-matrix representing the following graph:



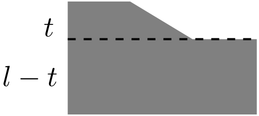
- In all cases, $\epsilon_m^* = 0$.

(l, r) SC-LDPC Protographs With Random Access

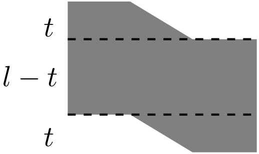
Start: B



Cut: B_1



Paste: $\binom{B_1}{B_2}$



(l, r, t) SC-LDPCL Construction

- Let $t \in \{1, 2, \ldots, l-2\}$, and let A_1 be a $t \times r$ matrix given by

$$A_1 = \begin{pmatrix} \underline{1} & & & & & \underline{0} \\ & & & \text{\huge 0} & & \\ \underline{1} & \underline{1} & & & & \underline{0} \\ \underline{1} & \underline{1} & \underline{1} & & & \underline{0} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \underline{1} & \underline{1} & \underline{1} & \cdots & \underline{1} & \underline{0} \end{pmatrix},$$

  where $\underline{1}$ and $\underline{0}$ are length-$\lfloor \frac{r}{t+1} \rfloor$ all-one vector and length-$\left(r - t\lfloor \frac{r}{t+1} \rfloor\right)$ all-zero vector, respectively.

- Let $A_2 = 1^{(l-t)\times r}$ be an all-ones matrix.

Place M copies of $(B_1; B_2)$ on the diagonal, where $B_1 = (A_1; A_2)$ and $B_2 = 1^{l\times r} - B_1$.

(l, r, t) SC-LDPCL Protographs with $M = 3$ Sub-Blocks

Example: the $(4, 8, 1)$ SC-LDPCL protograph.

$$\left(\begin{array}{cccccccc||cccccccc||cccccccc}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1
\end{array}\right)$$

(l, r, t) SC-LDPCL Protographs with $M = 3$ Sub-Blocks

Example: the $(4, 8, 2)$ SC-LDPCL protograph.

$$\left(\begin{array}{cccccccc||cccccccc||cccccccc}
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
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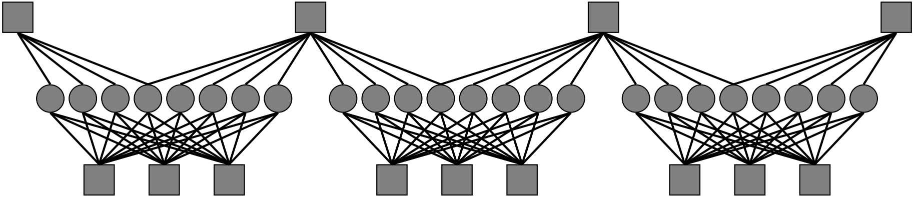
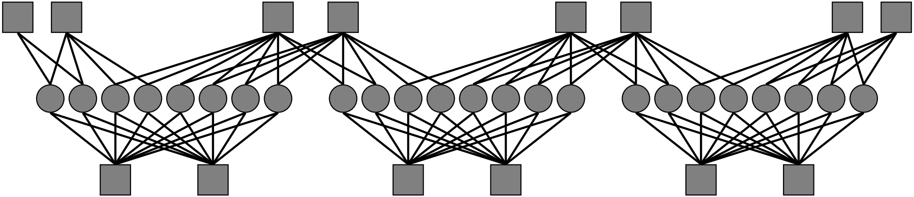
(l, r, t) SC-LDPCL Protographs with $M = 3$ Sub-Blocks

$(l=4, r=8, t=1)$



$(l=4, r=8, t=2)$



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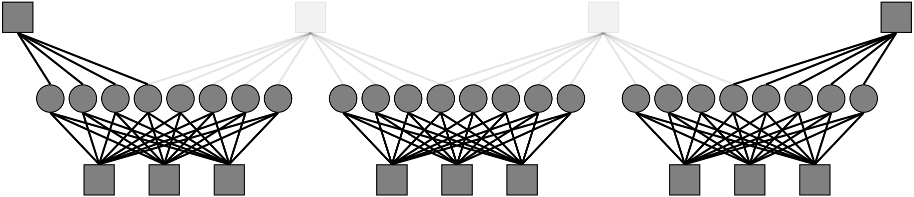
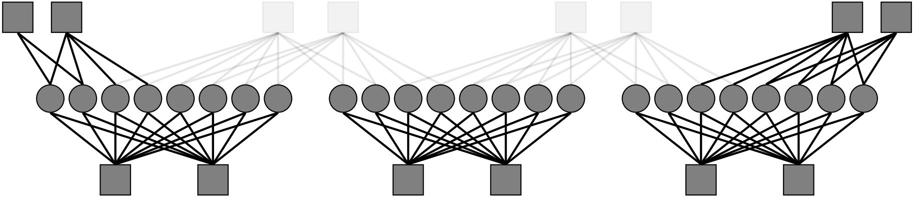


$(l = 4, r = 8, t = 2)$



Theorem

Let $1 < M$, $2 < l < r$ and $t \in \{1, 2, \ldots l-2\}$. Then, every (l, r, t) SC-LDPCL protograph with M sub-blocks has sub-block random access ability: for every $m \in \{1, 2, \ldots, M\}$, $\epsilon_m^* \neq 0$.

- For fast access, one can locally decode a target sub-block.
  - Local threshold of sub-block m: ϵ_m^*.
  - Lower than the worst-case requirements.
- In extreme error events, the entire protograph is decoded.
  - Global threshold: ϵ_G^*.
  - Provides the data-reliability needed.

t As a Design Tool

$t \in \{1, 2, \ldots, l-2\}$ controls the sub-block vs. entire-block trade-off.

Table: The $(5, 10, t)$ SC-LDPCL protographs with $M = 6$ sub-blocks.

t	ϵ_1^*	$\epsilon_2^*, \ldots, \epsilon_5^*$	ϵ_6^*	ϵ_G^*
1	0.3667	0.3079	0.3667	0.3734
2	0.4017	0.2538	0.3935	0.4148
3	0.3333	0.1111	0.4263	0.4654

- $t = 0$: sub-blocks are fully localized – no coupling.
- $t = l-1$: sub-blocks are fully coupled – no local decoding.

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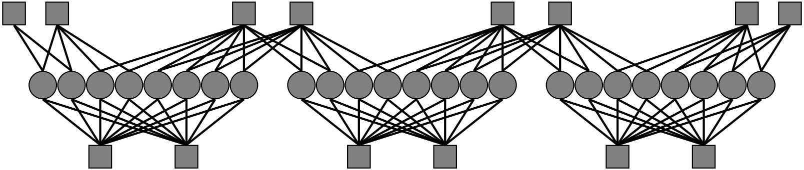
- The terminating sub-blocks are more reliable due to terminating checks.
- If we successfully decode a sub-block, its neighbors become more reliable!



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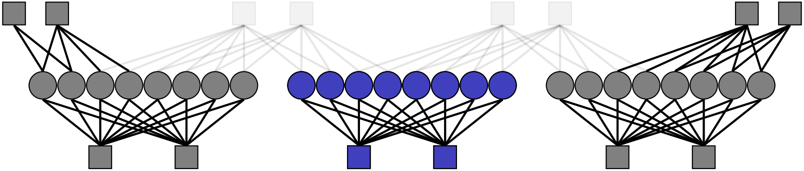

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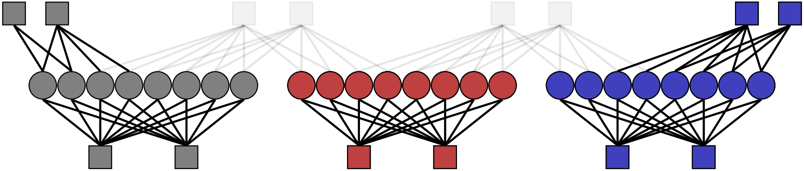

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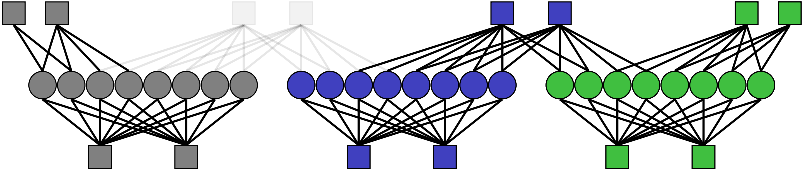

Table: The $(5, 10, t)$ SC-LDPCL protographs with $M = 6$ sub-blocks.

t	ϵ_1^*	$\epsilon_2^*, \ldots, \epsilon_5^*$	ϵ_6^*
1	0.3667	0.3079	0.3667
2	0.4017	0.2538	0.3935
3	0.3333	0.1111	0.4263

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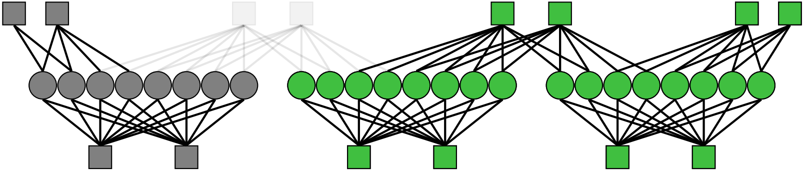


A Practical Channel for Storage Devices

- In many storage devices, different sub-blocks suffer different channels.
- We consider a family of channels with memory: the channel parameter changes between sub-blocks (McEliece and Stark, 1984).

The Sub-Block Varying-Erasure Channel

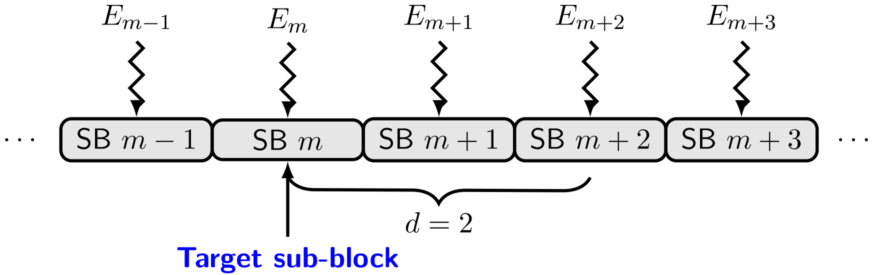
Let $E_1, E_2, \ldots, E_M$ be i.i.d random variables taking values in $[0, 1]$. In the **sub-block varying erasure channel**, the bits of sub-block $m \in \{1, 2, \ldots, M\}$ are transmitted over the channel $BEC(E_m)$.

Semi-Global Decoding

Semi-Global Decoding

Decode sub-block $m+d$ down to sub-block m in a sequential fashion:

- For every $j \in \{1, 2, \ldots, d\}$, use the information decoded from sub-block $m+j$, to decode sub-block $m+j-1$.



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Semi-Global Decoding

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- For every $j \in \{1, 2, \ldots, d\}$, use the information decoded from sub-block $m+j$, to decode sub-block $m+j-1$.

- Semi-global decoding is highly motivated by:
  - Random-access abilities of SC-LDPCL codes.
  - Coupling in SC-LDPCL codes: sub-blocks can help neighbor sub-blocks.
  - Channels with sub-block varying parameters.

Proposition

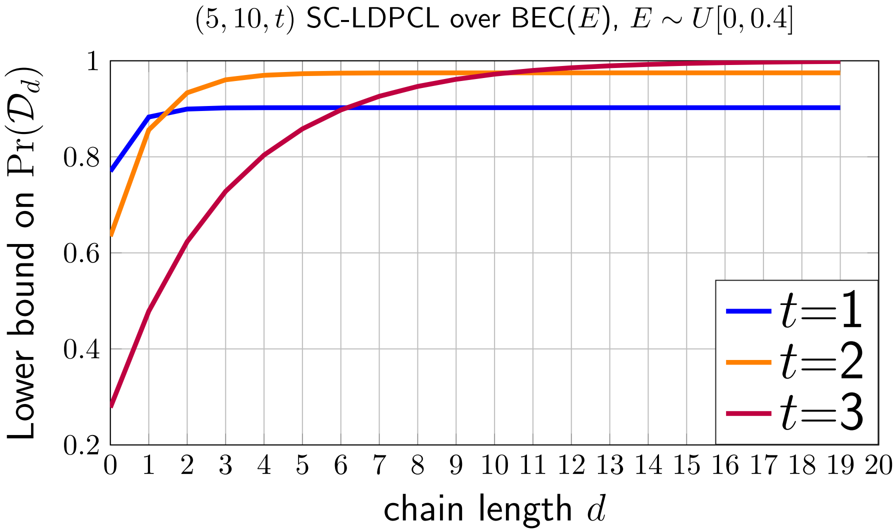
Consider **semi-global decoding** over the sub-block varying-erasure channel $BEC(E)$. Let

- ϵ_L be the sub-block threshold in a SC-LDPCL protograph.
- ϵ_S be the sub-block threshold **given that the neighbor sub-block was successfully decoded**.
- $P_L = \Pr(E < \epsilon_L),\ P_S = \Pr(E < \epsilon_S),\ q = P_S - P_L$.
- $\Pr(\mathcal{D}_d)$ be the probability of successful length-d semi-global decoding.

Then,

$$\Pr(\mathcal{D}_d) \geq P_L \frac{1-q^{d+1}}{1-q}.$$

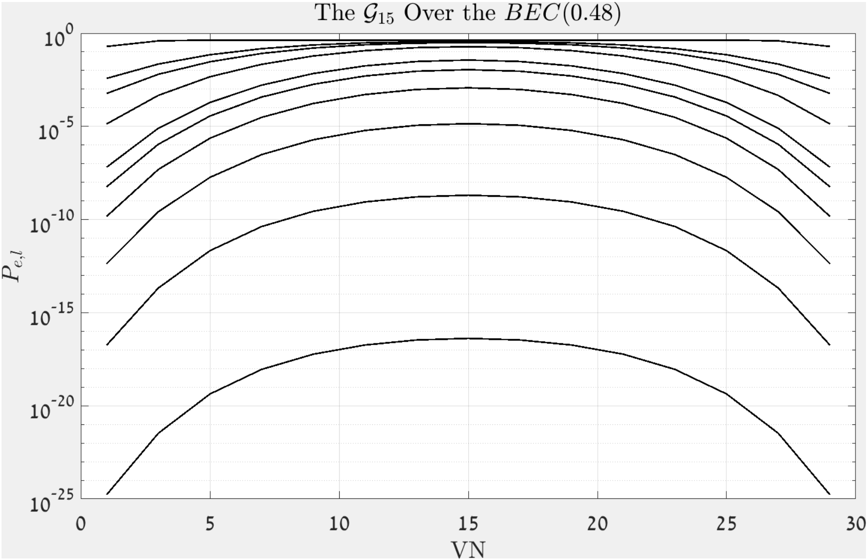
Example

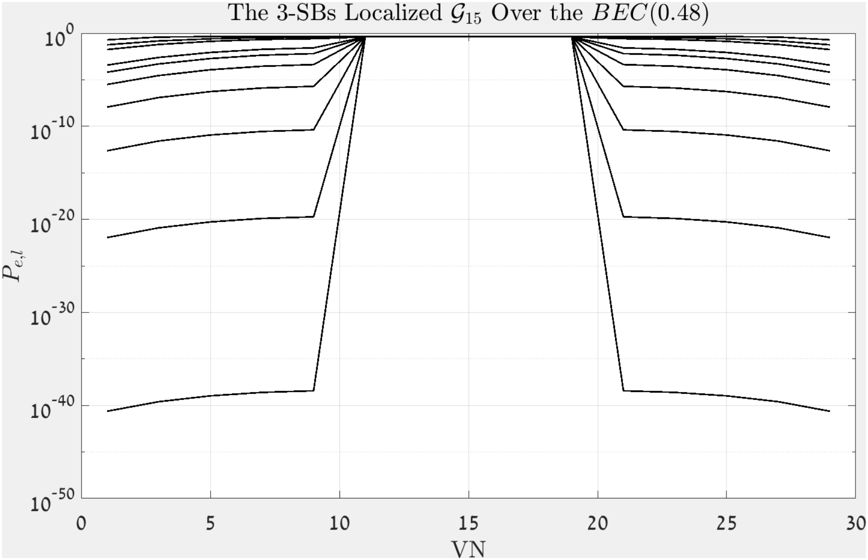

$(5, 10, t)$ SC-LDPCL over BEC(E), $E \sim U[0, 0.4]$
Lower bound on $\Pr(\mathcal{D}_d)$
chain length d
0.2
0.4
0.6
0.8
1
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
t=1
t=2
t=3

Open Problems

- Improving local threshold via irregular (local) graphs.
- Constructing more random-access codes, e.g., polar codes.

Thank You!


The $\mathcal{G}_{15}$ Over the $BEC(0.48)$
10^{0}
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
$P_{e,l}$
0
5
10
15
20
25
30
VN



The 3-SBs Localized $\mathcal{G}_{15}$ Over the $BEC(0.48)$
10^{0}
10^{-10}
10^{-20}
10^{-30}
10^{-40}
10^{-50}
$P_{e,l}$
0
5
10
15
20
25
30
VN